PLANE NETS WITH EQUAL INVARIANTS.*

By Gabriel M. Green.

Two geometric characterizations of plane nets with equal Laplace invariants have been given by Koenigs. One of them, which identifies such a net as the projection, upon a plane, of the asymptotic net on a curved surface, has no analogue in the corresponding case of a conjugate net with equal invariants on a surface.* The other characterization, however, which involves certain conics connected with the net, is essentially the same for a plane net and for a conjugate net on a surface.† Recently, an elegant characterization of conjugate nets with equal invariants was given by Wilczynski,† according to which the developables of the congruence of lines joining the first and minus first Laplace transforms of the points of the surface correspond to a conjugate net on the surface. This characterization can not, of course, be applied to plane nets, because there is no relation in the plane to correspond to that of conjugate directions on a surface.

Another characterization of conjugate nets with equal invariants has been given by the present writer,‡ viz., that for such a net each line of the congruence of lines joining the first and minus first Laplace transforms is met in its focal points by the tangents to the curves which correspond to the developables of the congruence. It is our purpose to extend this characterization to planar nets. Naturally, we can not speak of the developables of a congruence of lines in a plane, since every one-parameter family of lines in a plane forms a developable; therefore our characterization can not be carried over to the plane case in its entirety.

Let

$$y^{(k)} = y^{(k)}(u, v) \qquad (k = 1, 2, 3) \tag{1}$$

be the point equations, in homogeneous coördinates, of a net of curves in the plane. We suppose as usual that through every point y of a certain region of the plane pass two and only two curves of the net, the curves

* Presented to the American Mathematical Society, September 5, 1917.

† "The general theory of congruences," Transactions of the American Mathematical Society, vol. 16 (1915), pp. 311–327.

‡ "Projective differential geometry of one-parameter families of space curves, and conjugate nets on a curved surface," American Journal of Mathematics, vol. 38 (1916), pp. 287–324. Cf. in particular p. 318. Cf. also a note on this theorem in the Bulletin of the American Mathematical Society, vol. 24 (1918), pp. 221–225.

$u = \text{const.}$ and $v = \text{const.}$, and we shall suppose moreover that these curves have distinct tangents at y. Then the three functions y are such that the determinant

$$\begin{vmatrix} y_u^{(1)} & y_v^{(1)} & y^{(1)} \\ y_u^{(2)} & y_v^{(2)} & y^{(2)} \\ y_u^{(3)} & y_v^{(3)} & y^{(3)} \end{vmatrix}$$

is different from zero throughout the region under consideration, and the said three functions y are a fundamental set of solutions of a completely integrable system of partial differential equations of the form*

$$\begin{aligned} y_{uu} &= a y_u + b y_v + c y, \\ y_{uv} &= a' y_u + b' y_v + c' y, \\ y_{vv} &= a'' y_u + b'' y_v + c'' y. \end{aligned} \tag{2}$$

Any sufficiently regular one-parameter family of straight lines in the plane has an envelope. Under the suppositions already made, the tangents to the curves $v = \text{const.}$, along a fixed curve $u = \text{const.}$, have an envelope, and the point of this envelope which corresponds to y is

$$\rho = y_u - b'y. \tag{3}$$

Similarly, the tangents to the curves $u = \text{const.}$ along a fixed curve $v = \text{const.}$ have an envelope, and the point of this envelope corresponding to y is†

$$\sigma = y_v - a'y. \tag{4}$$

We shall call the line $\rho\sigma$ the *ray* of the point y, borrowing the term used by Wilczynski for the line joining the minus first and first Laplace transforms of a conjugate net on a curved surface. In the present case, the second of equations (2) is of the Laplace type, and ρ and σ are its minus first and first Laplace transforms respectively.

We shall denote by N the particular planar net under consideration. Let N' be any other net superposed on N. It may of course be defined by the equations

$$\varphi(u, v) = c_1, \qquad \psi(u, v) = c_2,$$

where the Jacobian of φ and ψ does not vanish anywhere in the region under consideration. Let t_1 and t_2 denote the tangents at y to the two curves of the net N, and t_1' and t_2' the tangents at y to the two curves of the net N'. The harmonic conjugate of t_1' with respect to t_1 and t_2 is a

* E. J. Wilczynski, "One-parameter families and nets of plane curves," Transactions of the American Mathematical Society, vol. 12 (1911), pp. 473–510.

† Cf. Wilczynski, loc. cit., p. 486.

definite line through y, say T_1, and the harmonic conjugate of t_2' with respect to t_1 and t_2 is a definite line T_2 through y and distinct from T_1.

Now allow the point y to trace out a curve C_1' of the net N'. Then the corresponding rays $\rho\sigma$ envelop a curve, and the ray $\rho\sigma$ which corresponds to a particular point y is tangent to this envelope at a particular point F_1. Similarly, the other curve C_2' of the net N' will yield a second point F_2 on the same line $\rho\sigma$. On each line $\rho\sigma$, we have therefore determined two points F_1 and F_2, which we shall call the *foci* or *focal points of the ray with respect to the net* N'. If t_1', t_2' are the tangents at y to the curves C_1', C_2' respectively, we shall say that F_1, F_2 correspond to t_1', t_2' respectively. Of course the net N' has thus far been quite arbitrary. We shall show presently that it is always possible to choose it in a unique way so that it shall have the following relation to the net N: *the lines* T_1 *and* T_2 *defined above are to meet the corresponding ray in the respective focal points* F_1 *and* F_2 *of the ray.* We shall say that the net N' is then *congruentially associated* with the net N, and shall refer to its defining property as the *congruential property*, because of the analogy to the case of the congruence of rays belonging to a conjugate net on a curved surface. The net N' is in this latter case the net of curves on the surface which correspond to the developables of the ray congruence.

The geometric characterization which we wish eventually to establish may now be stated as follows: *A necessary and sufficient condition that a planar net have equal invariants is that at each point of the net the tangents to the congruentially associated net meet the corresponding ray in the focal points of the ray, each tangent meeting the ray in the focal point which corresponds to the other.*

We shall first establish the existence of a unique net N' which is congruentially associated with the given net N. Let us suppose that a curve of the net N' is defined by means of the equations

$$u = u(t), \qquad v = v(t),$$

and let us denote the curve by C_t, putting in evidence the parameter t along the curve. If y is a point of the curve C_t, a point on the tangent to C_t at y is given by

$$y_u \frac{du}{dt} + y_v \frac{dv}{dt}.$$

That is, the tangent to C_t is the line joining y to this point. Of course, the actual values of du/dt and dv/dt are not significant in determining this tangent, but only the ratio dv/du, which we shall denote by μ.

As the point y traces out the curve C_t, the corresponding ray $\rho\sigma$ envelops a curve, and we wish to determine the corresponding point of

this curve. It is a point on the line $\rho\sigma$, and is therefore given by an expression of the form

$$R = \rho + \lambda\sigma,$$

where λ is to be determined so as to make R a focus of the ray. Let E_t denote the envelope; then the tangent to E_t is the line which joins the point R to the point

$$R_u \frac{du}{dt} + R_v \frac{dv}{dt},$$

or, what is the same thing, to the point $R_u + \mu R_v$, where $\mu = dv/du$. Now this line is to coincide with the line $\rho\sigma$, since $\rho\sigma$ is tangent to its envelope. Therefore we seek the condition that the three points ρ, σ, $R_u + \mu R_v$ lie on a line. We have, on differentiating (3) and (4) and using the values of the second derivatives of y taken from (2),

$$\rho_u = (a - b')y_u + by_v + (c - b_u')y, \qquad \rho_v = a'y_u + (c' - b_v')y,$$

$$\sigma_u = b'y_v + (c' - a_u')y, \qquad \sigma_v = a''y_u + (b'' - a')y_v + (c'' - a_v')y.$$

On replacing y_u and y_v by their values in terms of ρ, σ, and y, we obtain finally

$$\rho_u = (a - b')\rho + b\sigma + \mathfrak{C}y, \qquad \rho_v = a'\rho + Ky,$$

$$\sigma_u = b'\sigma + Hy, \qquad \sigma_v = a''\rho + (b'' - a')\sigma + \mathfrak{C}''y,$$

where

$$\mathfrak{C} = c + b'(a - b') + a'b - b_u', \qquad \mathfrak{C}'' = c'' + a'(b'' - a') + a''b' - a_v',$$

and where

$$H = c' + a'b' - a_u', \qquad K = c' + a'b' - b_v'$$

are the *Laplace-Darboux invariants* of the net. We may now calculate $R_u + \mu R_v$ in terms of ρ, σ, and y, and obtain without difficulty

$$R_u + \mu R_v = A\rho + B\sigma + (\mathfrak{C}''\lambda\mu + H\lambda + K\mu + \mathfrak{C})y,$$

where the values of A and B do not concern us. If this point is to lie on the line $\rho\sigma$, the coefficient of y must vanish, since y, ρ, σ can never lie on the same line; i. e.,

$$\mathfrak{C}''\lambda\mu + H\lambda + K\mu + \mathfrak{C} = 0. \tag{5}$$

Since μ is known for a given net N', the value of λ is found immediately from equation (5), and then the point $R = \rho + \lambda\sigma$ is the corresponding focus of the ray $\rho\sigma$. But we wish this point to coincide with the point in which $\rho\sigma$ is met by the line T which is the harmonic conjugate of the tangent to C_t with respect to the tangents of the net N, in order that N' may be congruentially associated with N. The condition for this is

that $\lambda = -\mu$. Substituting this value for λ in equation (5), and replacing μ by its value dv/du, we find that *the differential equation of the net N' which is congruentially related to the given net N is*

$$\mathfrak{C}''du^2 + (H - K)dudv - \mathfrak{C}dv^2 = 0. \tag{6}$$

The process by which we obtained this equation shows that the net N' is unique. Now, if the net N has equal invariants, i. e., $H = K$, the middle term of equation (6) is absent. We may therefore state the theorem:

A necessary and sufficient condition that the net N have equal invariants is that the two tangents to the curves of the congruentially associated net N' at any point y separate harmonically the two tangents to N at y.

This characterization is of course equivalent to the one announced above, and reduces to it immediately when the definition of the congruentially associated net is applied.

A number of interesting questions suggest themselves in the present connection; an important one is, whether or not the relation between a net N and its congruentially associated net N' is a reciprocal one, and, if it is not, under what conditions it is. Again, the present writer's recent researches in the general theory of surfaces and rectilinear congruences* indicate a close analogy between the given net N and the asymptotic net on a curved surface, especially as regards the congruential property.

An isothermal planar net being an orthogonal net with equal Laplace-Darboux invariants, it is obvious that our theorem affords a purely geometric characterization of such nets. We have elsewhere given an analogous characterization of isothermal nets on a curved surface.†

HARVARD UNIVERSITY,
August 23, 1917.

* To appear in the Transactions of the American Mathematical Society.

† Transactions of the American Mathematical Society, vol. 18 (1917), pp. 480–488.